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Model reduction for state-space symmetric systems [1]

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Abstract

In this paper, the model reduction problem for state-space symmetric systems is investigated. First, it is shown that several model reduction methods, such as balanced truncation, balanced truncation which preserves the DC gain, optimal and suboptimal Hankel norm approximations, inherit the state-space symmetric property. Furthermore, for single input and single output (SISO) state-space symmetric systems, we prove that the H_∞ norm of its transfer functions can be calculated via two simple formulas. Moreover, the SISO state-space symmetric systems are equivalent to systems with zeros interlacing the poles (ZIP) under mild conditions. © 1998 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

Many physical phenomenon can be described by using some form of mathematical modeling. In designing a controller for a physical system, this step is very important. In many engineering problems, such as in robotics and aeronautics, the model that results is complex and usually of high order. Such high-order models will result in high-order controllers and along with the additional computational burdens associated, the complexities of the model and controller may make simulation and physical implementation impossible. As well as the potential difficulties, intuitive understanding may also be lost, and engineers may no longer enjoy a feel for the system, with such knowledge buried beneath mountains of equations. The need for low-order models is thus clear [1].

It is important when obtaining a low-order model, not to sacrifice vital characteristics of the physical system, such as stability, transient response, steady-state error and so on. Different methods of obtaining reduced-order models with these characteristics, have been presented over the last 30 years, each focusing primarily on some properties of the system which has been deemed important, such as balanced truncation [5], optimal Hankel norm approximation [8] and balanced truncation preserving DC gain [10]. Until now, the properties inherited from original systems by the reduced systems have not been discussed in depth. Recently, model reduction for SISO systems with ZIP property has been discussed in [14] and it is shown that many model reduction methods can inherit the ZIP property. In this paper, we study the model reduction problem for state-space symmetric systems which appear very often in circuit systems [11,2]. This paper extends the results reported in [14] substantially.

An important motivation for this paper can be found in [5] or [9]. It has been shown there that for a very

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special case of ZIP system (a special kind of symmetric system), the bounds on the H_∞ norm and the model reduction error while performing balanced truncation are tight. This motivates us to study the bounds on H_∞ norm and model reduction error for state-space symmetric systems. Another motivation is the extensive study undertaken by many researchers on the properties on the symmetric systems [4, 6, 12].

The layout of this paper is as follows: In Section 2, some basic results which are useful and interesting are presented. In Section 3, our main results are developed. Section 4 will establish a relationship between state-space symmetric system and ZIP systems for SISO case. Conclusions are given in Section 5.

2. Basic results

Definition 2.1. A system $G(s)$ is said to be state-space symmetric iff there exists a minimal state-space realization (A,B,C) satisfying

$$A=A^{\mathrm T},\qquad C=B^{\mathrm T} \tag{1}$$

where T denotes transpose of a matrix.

A system $G(s)$ is said to be symmetric if $G(s)=G^{\mathrm T}(s)$ [12]. It is clear that a state-space symmetric system must be a symmetric system, the converse is not true generally. For discussions related to state space realizations of symmetric systems, see [3, 7, 13]. It is proved in [13] that every symmetric system admits a balanced realization which is parity symmetric.

Definition 2.2. A SISO system

$$G(s)=\frac{\prod_{j=1}^{n-1}(s-z_j)}{\prod_{i=1}^{n}(s-a_i)}$$

is a ZIP iff $a_i<z_i<a_{i+1}$ holds for $i=1,2,\ldots,n-1$.

The state-space symmetric systems in Definition 2.1 are for multi-input–multi-output (MIMO) systems and systems with ZIP property in Definition 2.2 are only for SISO systems. It should be noted that ZIP systems can also be defined for MIMO systems [15]. Another issue in Definition 2.1 is that we do not need all minimal realizations satisfying Eq. (1), which implies that symmetry is system's own property independent of its state-space realizations. In this section, we will present some basic results on these two types of systems in the form of theorems and lemmas.

Theorem 2.1. *For any stable state-space symmetric realizations, all the three Gramians are equal*

$$P=Q=R, \tag{2}$$

where

$$P=\int_0^{+\infty} \mathrm{e}^{At}BB^{\mathrm T}\mathrm{e}^{A^{\mathrm T}t}\,\mathrm{d}t$$

is the Controllability Gramian,

$$Q=\int_0^{+\infty} \mathrm{e}^{At}C^{\mathrm T}C\mathrm{e}^{A^{\mathrm T}t}\,\mathrm{d}t$$

is the Observability Gramian and

$$R=\int_0^{+\infty} \mathrm{e}^{At}BC\mathrm{e}^{A^{\mathrm T}t}\,\mathrm{d}t$$

is the Cross Gramian.

The proof follows immediately from the definitions of the Gramians.

Theorem 2.2. *Given a stable state-space symmetric system $G(s)$ with minimal realization (A,B,C) satisfying Eq.* (1). *Assume that (A_b,B_b,C_b) is any balanced realization with*

$$A_b=TAT^{-1},\quad B_b=TB,\quad C_b=CT^{-1}.$$

Then the following hold:

(i) $T^{\mathrm T}=T^{-1}$.

(ii) (A_b,B_b,C_b) *is also a state-space symmetric system.*

The proof is straightforward and omitted here.

This theorem revealed the relationship between a given state-space system and its corresponding balanced realizations. In detail, for any balanced realizations obtained from the given realization (A,B,C), its transformation matrix T will satisfy the equation in (i). Furthermore, its corresponding balanced realization is also state-space symmetric.

Lemma 2.1. *$G(s)$ is a state-space symmetric system iff it has a diagonal realization $\{A,B,C\}$ with $A=\mathrm{diag}(a_1,a_2,\ldots,a_n)$ and $B^{\mathrm T}=C$*

This lemma is an obvious fact and its proof is omitted. The next two lemmas are from [6] for SISO systems, which will be used in sequel.

Lemma 2.2. *The following statements are equivalent*:

(i) $G(s)$ *is a stable strictly proper ZIP system.*
(ii) $G(s)$ *can be written as*

$$G(s)=\sum_{i=1}^{n}\frac{b_i}{s+a_i},\quad a_i>0,\ b_i>0,\qquad a_i\neq a_j,\ i\neq j. \tag{3}$$

(iii) $G(s)$ *has a diagonal realization*

$$\left(\begin{array}{c|c} A & b \\ \hline c & 0\end{array}\right)\quad \text{with } A=\operatorname{diag}(-a_1,-a_2,\ldots,-a_n)$$

and $b^{\mathrm{T}}=c=[b_1^{1/2},b_2^{1/2},\ldots,b_n^{1/2}]$.

Lemma 2.3. *Let X be a definite matrix (positive or negative) and Y a symmetric matrix. Assume that V is the eigenvector matrix of XY. Then both $V^{\mathrm{T}}YV$ and $V^{\mathrm{T}}X^{-1}V$ are diagonal.*

From Lemma 2.2, we can see that ZIP system is a special case of state-space symmetric system. We will discuss more the relationship between the two systems in Section 4.

3. Main results

In this section, we will study the symmetric property inherited from original systems by the reduced-order model obtained by using several model reduction methods, i.e., balanced truncation, balanced truncation preserving DC gain, optimal and suboptimal Hankel norm approximations.

Theorem 3.1. *Reduced-order model realizations obtained using balanced truncation method for stable state-space symmetric systems are also state-space symmetric.*

Proof. Let any balanced realization be partitioned as

$$A_b=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix},\qquad B_b=\begin{bmatrix}B_1\\ B_2\end{bmatrix},$$
$$C_b=[C_1,C_2]. \tag{4}$$

From part (ii) of Theorem 2.2, (A_b,B_b,C_b) is a state-space symmetric system which implies that $A_{11}=A_{11}^{\mathrm{T}}$ and $B_1=C_1^{\mathrm{T}}$. This shows that (A_{11},B_1,C_1) is also a state-space symmetric system. □

Theorem 3.2. *Reduced-order model realizations obtained using balanced truncation method which preserves the DC gain for stable state-space symmetric systems are also state space symmetric systems.*

Proof. Let the system matrices be partitioned as in Eq. (4). The balanced truncation method which preserves the DC gain gives the following reduced model:

$$\left(\begin{array}{c|c} A_{11}-A_{12}A_{22}^{-1}A_{12} & B_1-A_{12}A_{22}^{-1}B_1 \\ \hline C_1-C_2A_{22}^{-1}A_{12}^{\mathrm{T}} & -C_2A_{22}^{-1}B_2\end{array}\right). \tag{5}$$

Recalling that $A_{12}=A_{21}^{\mathrm{T}}$, $C_1=B_1^{\mathrm{T}}$ and $C_2=B_2^{\mathrm{T}}$, we can easily see that Eq. (5) is also a state-space symmetric system. □

In order to prove the result for optimal Hankel norm approximation, we need the following preparatory results.

Theorem 3.3. *Given a stable state-space symmetric system $G(s)$ with balanced realization (A,B,C). Assume that its controllability and observability Gramians are $P=Q=\operatorname{diag}\{\sigma_1,\sigma_2,\ldots,\sigma_n)$ with decreasing order*

$$\sigma_1\geqslant\sigma_2\geqslant\cdots\geqslant\sigma_k>\sigma_{k+1}$$
$$=\cdots=\sigma_{k+r}>\sigma_{k+r+1}\geqslant\cdots\geqslant\sigma_n.$$

Denote

$$\Sigma=\operatorname{diag}(\sigma_1,\sigma_2,\ldots,\sigma_k,\sigma_{k+r+1},\ldots,\sigma_n,\sigma_{k+1},\ldots,\sigma_{k+r})$$
$$=\operatorname{diag}(\Sigma_1,\sigma_{k+1}I_r) \tag{6}$$

and

$$S=\begin{pmatrix}I_k & 0\\ 0 & -I_{n-k-r}\end{pmatrix}.$$

Let

$$\hat G(s)=\left(\begin{array}{c|c} \hat A & \hat B \\ \hline \hat C & \hat D\end{array}\right)$$

be a balanced system and $(G-\hat G)$ be all pass with $\| G-\hat G\|_\infty=\sigma_{k+1}$. Further, assume that the stable *and unstable parts of* $\hat{G}(s)$ *are*

$$\hat{G}^{+}(s)=\left(\begin{array}{c|c}\hat{A}^{+} & \hat{B}^{+}\\ \hline \hat{C}^{+} & \hat{D}^{+}\end{array}\right),$$

$$\hat{G}^{-}(s)=\left(\begin{array}{c|c}\hat{A}^{-} & \hat{B}^{-}\\ \hline \hat{C}^{-} & 0\end{array}\right).$$

Then the following hold:
(i) $\hat{A}S=(\hat{A}S)^{\mathrm{T}}$ *is negative definite*;
(ii) $\hat{B}^{\mathrm{T}}=\hat{C}S$;
(iii) $(\hat{B}^{+})^{\mathrm{T}}=\hat{C}^{+}$.

Proof. In order to construct an optimal Hankel norm approximation, rows and columns corresponding to the singular value σ_{k+1} need to be deleted. Assume the remaining system matrices after deleting are given by (A_1,B_1,C_1) and Σ_1. The all-pass dilation denoted by

$$G_d(s)=\left(\begin{array}{c|c}A_d & B_d\\ \hline C_d & D_d\end{array}\right)$$

can be obtained using the following matrices:

$$P_1=\begin{pmatrix}\Sigma_1 & -\Gamma\\ -\Gamma & \Gamma\Sigma_1\end{pmatrix},\quad Q_1=\begin{pmatrix}\Sigma_1 & I\\ I & \Gamma^{-1}\Sigma_1\end{pmatrix},$$

$$\Gamma=\Sigma_1^2-\sigma_{k+1}^2 I.$$

From the all-pass properties [8], we can conclude that $D_d=\sigma_{k+1}I$. Also from the property

$$D_d^{\mathrm{T}}[C_1,-C_d]+[B_1^{\mathrm{T}},B_d^{\mathrm{T}}]Q_1=0$$

and noting from Theorem 2.1 that $B_1^{\mathrm{T}}=C_1$, we obtain

$$B_d=-B_1^{\mathrm{T}}(\Sigma_1+\sigma_{k+1}I),$$
$$C_d=-C_1(\Sigma_1+\sigma_{k+1}I)\Gamma^{-1}. \tag{7}$$

From the augmented Lyapunov equation, one can get

$$A_d=(\sigma_{k+1}^2A_1^{\mathrm{T}}+\Sigma_1A_1\Sigma_1-\sigma_{k+1}C_1^{\mathrm{T}}C_1)\Gamma^{-1}S. \tag{8}$$

Let Γ_1 be the absolute value of Γ, i.e., $\Gamma_1=\Gamma S$. The balancing similarity transformation, which leads to $\Sigma_b=\Sigma_1S$ is given by $\Gamma_1^{1/2}$. Performing such a transformation, one obtains

$$\hat{A}=\Gamma_1^{-1/2}(\sigma_{k+1}^2A_1+\Sigma_1A_1\Sigma_1-\sigma_{k+1}C_1^{\mathrm{T}}C_1)\Gamma_1^{-1/2}S, \tag{9}$$

$$\begin{aligned}\hat{B}^{\mathrm{T}}&=-B_1^{\mathrm{T}}(\Sigma_1+\sigma_{k+1}I)\Gamma_1^{-1/2},\\ \hat{C}&=-C_1(\Sigma_1+\sigma_{k+1}I)\Gamma_1^{-1/2}S.\end{aligned} \tag{10}$$

Since A is stable, $A_1=A_1^{\mathrm{T}}$ is also stable. Then it is easy to see that $\hat{A}S$ is the sum of three negative-definite matrices and thus part (i) is proved. Noting $B_1=C_1^{\mathrm{T}}$, we can obtain part (ii) easily. Next, we prove part (iii). Let V be the eigenvectors of $\hat{A}$. Note that $\hat{A}=(\hat{A}S)S$ with $\hat{A}S$ being negative definite and S symmetric. Then from Lemma 2.3, $A_f=V^{-1}(\hat{A}S)V^{-\mathrm{T}}$ and $S_f=V^{\mathrm{T}}SV$ are diagonal. Moreover, the diagonal representation of $\hat{A}$ will be $\hat{A}_d=V^{-1}\hat{A}V=A_fS_f$. The diagonalizing transformation leads to $\hat{B}_d=V^{-1}\hat{B}$ and $\hat{C}_d=\hat{C}V$. Recalling that $\hat{B}^{\mathrm{T}}=\hat{C}S$, one obtains $\hat{B}_d=\hat{C}_dS_f^{-1}$.

Since $\hat{A}S$ is negative definite, A_f is also negative definite. The signs of the entries in the diagonal of S_f partition the stable and unstable parts of $\hat{A}_d$. If $(S_f)_{i,i}>0$, the ith mode of $\hat{A}_d$ is stable. Since $\hat{A}S$ is negative definite with S being in a special form, it is routine to prove $V^{\mathrm{T}}=V^{-1}$ and all the entries in S_f are ±1. Now, construct the stable parts as follows. Let $\hat{C}^{+}$ be the columns corresponding to the diagonal entries $+1$ and $\hat{B}^{+\mathrm{T}}=\hat{C}^{+}$. Thus (iii) is true. □

Theorem 3.4. (i) *Optimal Hankel norm approximation of a stable state-space symmetric system is also a state-space symmetric system.*

(ii) *Suboptimal Hankel norm approximation of a stable state-space symmetric system is also a state-space symmetric systems.*

Proof. (i) Note $\hat{G}^{+}(s)$ in Theorem 3.3 is an optimal Hankel norm approximate and from part (iii) of Theorem 3.3, $(\hat{B}^{+})^{\mathrm{T}}=\hat{C}^{+}$ with $\hat{A}^{+}$ in a diagonal representation, $\hat{G}^{+}(s)$ is state-space symmetric.

(ii) When suboptimal Hankel norm approximation is performed, there is no removal of rows and columns, all other procedures in Theorem 3.3 are still kept the same. It is easy to visualize that $\hat{G}^{+}(s)$ is still state-space symmetric. □

4. SISO state-space symmetric systems

In this section, we will study the SISO state-space symmetric systems. First, we show that its H_∞ bound is tight and then we show its relationship with ZIP systems.

Theorem 4.1. *Given a stable SISO symmetric system* $G(s)=c(sI-A)^{-1}b$ *with* $c=b^{\mathrm{T}}$ *and* $A=A^{\mathrm{T}}$. *Then the* H_∞ *norm of* $G(s)$ *can be calculated as shown* *below*:

(1) $$\| G(s) \|_\infty = 2 \sum_{i=1}^{n} \sigma_i, \tag{11}$$

where σ_i denotes the Hankel singular values of $G(s)$.

(2) $$\| G(s) \|_\infty = -cA^{-1}b. \tag{12}$$

Proof. (1) First, assume that $G(s)$ has a balanced realization (A, b, c) with $A=(a_{i,j})$ and $b=(b_1, b_2, \dots, b_n)^{\mathrm{T}}$, its corresponding Gramians are supposed to be $P = Q = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$. From Theorem 2.1, we can conclude that $A = A^{\mathrm{T}}$ and $c = b^{\mathrm{T}}$. Since the following Lyapunov equation

$$AP + PA + bb^{\mathrm{T}} = 0 \tag{13}$$

is satisfied, we get

$$2\sigma_i a_{i,i} + b_i^2 = 0, \quad \sigma_i = -\frac{b_i^2}{2a_{i,i}}. \tag{14}$$

Thus,

$$2\sum_{i=1}^{n} \sigma_i = -\sum_{i=1}^{n} \frac{b_i^2}{a_{i,i}}. \tag{15}$$

It is well known that $2\sum_{i=1}^{n} \sigma_i$ is an upper bound for the H_∞ norm of $G(s)$. The proof will be complete if it is shown that this bound can be achieved at some frequency. Since there exists V such that $A_d = V^{\mathrm{T}}AV$ is diagonal with $V^{\mathrm{T}} = V^{-1}$, then it is easy to prove that

$$|G(0)| = -\sum_{i=1}^{n} \frac{b_i^2}{a_{i,i}}$$

and hence the proof is complete for part one. Part 2 can be proved by direct computation. □

Remark 4.1. 1. It should be noted that part (i) of Theorem 4.1 is not true for MIMO state-space symmetric systems.

2. Part (ii) of Theorem 4.1 is still true for stable MIMO state-space symmetric system, which is stated as following.

Given a stable MIMO state-space symmetric system (A, B, C), then its H_∞ norm is given by

$$\| G(s) \|_\infty = \lambda_{\max}(-CA^{-1}B),$$

where $\lambda_{\max}(\ )$ denotes the maximal eigenvalue of a matrix.

The proof of this assertion is given in Appendix A.

Theorem 4.2. *A SISO stable state-space symmetric system $G(s)$ is a ZIP system if and only if $G(s)$ has distinct poles.*

Proof. The "only if" part is obvious due to Lemma 2.2. We only prove "if" part here. Since A is a stable symmetric matrix, it can be diagonalized by a transformation matrix V with $V^{\mathrm{T}} = V^{-1}$. So, we can assume $G(s)$ has a diagonal realization (A_d, b_d, c_d) with $A_d = \mathrm{diag}(a_1, a_2, \dots, a_n)$ and $b_d = (b_1, b_2, \dots, b_n)^{\mathrm{T}}$. Due to the fact that (A_d, b_d, c_d) is a minimal realization, we can have $b_i \neq 0$ and $a_i \neq a_j$ for $i \neq j$. In this case, we find

$$G(s) = \sum_{i=1}^{n} \frac{b_i^2}{s - a_i}.$$

Thus, it is equivalent to a ZIP system $(A_{zip}, b_{zip}, c_{zip})$ with

$$A_{zip} = A_d, \qquad b_{zip} = (|b_1|, |b_2|, \dots, |b_n|)^{\mathrm{T}},$$

$$c_{zip} = b_{zip}^{\mathrm{T}}. \qquad \square$$

Remark 4.2. It is implied from Theorem 4.2 that SISO state-space symmetric systems are actually ZIP systems if they have distinct poles. Thus, several properties of ZIP systems reported in [14] are also true for state-space symmetric systems.

5. Conclusion

In this paper, we have showed that models obtained using several model reduction methods inherit the symmetric property from original systems for state-space symmetric systems. For a SISO stable state-space symmetric system, it is shown that its H_∞ norm can be calculated analytically in two ways. Moreover, a SISO stable state-space symmetric system is actually a ZIP system if it has distinct poles. The results in this paper should be investigated for general symmetric systems, which is left as a suggestion for future research work.

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Appendix A. Proof of part (ii) in Remark 4.1

First, we present the following lemmas.

Lemma A1. *For any Hermite matrix* $A=A^{\mathrm{H}}\geqslant 0$, *the following holds*:

$$\max_{y^{\mathrm{T}}y\leqslant 1}\{y^{\mathrm{T}}Ay\}=\lambda_{\max}(A).$$

This proof is obvious.

Lemma A2. *For any* $A=A^{\mathrm{T}}$, *we have*

$$\|C(sI-A)^{-1}C^{\mathrm{T}}\|_\infty = \max_{y^{\mathrm{T}}y\leqslant 1}\|y^{\mathrm{T}}C(sI-A)^{-1}C^{\mathrm{T}}y\|_\infty. \tag{16}$$

Proof. We need to prove two points: one is to prove that for any vector y with $y^{\mathrm{T}}y\leqslant 1$, we have

$$\|y^{\mathrm{T}}C(sI-A)^{-1}C^{\mathrm{T}}y\|_\infty\leqslant\|C(sI-A)^{-1}C^{\mathrm{T}}\|_\infty,$$

the other point is to show that there exists a vector y_0 with $y_0^{\mathrm{T}}y_0\leqslant 1$ which will satisfy

$$\|y_0^{\mathrm{T}}C(sI-A)^{-1}C^{\mathrm{T}}y_0\|_\infty=\|C(sI-A)^{-1}C^{\mathrm{T}}\|_\infty.$$

We now prove the first point. According to the definition of H_∞ norm, we have

$$\|\{y^{\mathrm{T}}C(sI-A)^{-1}C^{\mathrm{T}}y\|_\infty^2 = \lambda_{\max_\omega}\{y^{\mathrm{T}}C(\mathrm{j}\omega I-A)^{-1}C^{\mathrm{T}} \times yy^{\mathrm{T}}C(-\mathrm{j}\omega I-A^{\mathrm{T}})^{-1}C^{\mathrm{T}}y\}.$$

Since $y^{\mathrm{T}}y\leqslant 1$, we have $yy^{\mathrm{T}}\leqslant I$. Therefore,

$$\{y^{\mathrm{T}}C(\mathrm{j}\omega I-A)^{-1}C^{\mathrm{T}}yy^{\mathrm{T}}C(-\mathrm{j}\omega I-A^{\mathrm{T}})^{-1}C^{\mathrm{T}}y\} \leqslant\{y^{\mathrm{T}}C(\mathrm{j}\omega I-A)^{-1}C^{\mathrm{T}}C(-\mathrm{j}\omega I-A^{\mathrm{T}})^{-1}C^{\mathrm{T}}y\}.$$

So

$$\lambda_{\max_\omega}\{y^{\mathrm{T}}C(\mathrm{j}\omega I-A)^{-1}C^{\mathrm{T}} \times yy^{\mathrm{T}}C(-\mathrm{j}\omega I-A^{\mathrm{T}})^{-1}C^{\mathrm{T}}y\} \leqslant\lambda_{\max_\omega}\{y^{\mathrm{T}}C(\mathrm{j}\omega I-A)^{-1} \times C^{\mathrm{T}}C(-\mathrm{j}\omega I-A^{\mathrm{T}})^{-1}C^{\mathrm{T}}y\} \leqslant\|C(sI-A)^{-1}C^{\mathrm{T}}\|_\infty.$$

Now, we are in a position to prove the second point. Without loss of generality, we can assume that

$$A=\mathrm{diag}(a_1,a_2,\ldots,a_n),\quad a_i<0$$

and C is full row rank. It is easy to see that

$$\|C(sI-A)^{-1}C^{\mathrm{T}}\|_\infty^2 = \lambda_{\max_\omega}\{C(\mathrm{j}\omega I-A)^{-1}C^{\mathrm{T}}C(-\mathrm{j}\omega I-A)^{-1}C^{\mathrm{T}}\}$$

and

$$C(\mathrm{j}\omega I-A)^{-1}C^{\mathrm{T}}=C\Lambda C^{\mathrm{T}}+\mathrm{j}C\Lambda_1C^{\mathrm{T}},$$

where

$$\Lambda=\mathrm{diag}\left(\frac{a_1}{\omega^2+a_1^2},\ldots,\frac{a_n}{\omega^2+a_n^2}\right)$$

and

$$\Lambda_1=\mathrm{diag}\left(\frac{\omega}{\omega^2+a_1^2},\ldots,\frac{\omega}{\omega^2+a_n^2}\right).$$

From Lemma A2, we obtain

$$\lambda_{\max_\omega}\{C(\mathrm{j}\omega I-A)^{-1}C^{\mathrm{T}}C(-\mathrm{j}\omega I-A)^{-1}C^{\mathrm{T}}\} = \max_{y^{\mathrm{T}}y\leqslant 1}\lambda_{\max_\omega}\{y^{\mathrm{T}}C(\mathrm{j}\omega I-A)^{-1} \times C^{\mathrm{T}}C(-\mathrm{j}\omega I-A)^{-1}C^{\mathrm{T}}y\}.$$

It is not hard to prove that

$$\max_{y^{\mathrm{T}}y\leqslant 1}\lambda_{\max_\omega}\{y^{\mathrm{T}}C(\mathrm{j}\omega I-A)^{-1} \times C^{\mathrm{T}}C(-\mathrm{j}\omega I-A)^{-1}C^{\mathrm{T}}y\} \leqslant\max_{y^{\mathrm{T}}y\leqslant 1}\{y^{\mathrm{T}}C(-A)^{-1}C^{\mathrm{T}}C(-A)^{-1}C^{\mathrm{T}}y\}$$

which indicates that the H_∞ norm is reached at $\omega=0$. Therefore,

$$\|C(sI-A)^{-1}C^{\mathrm{T}}\|_\infty^2=\lambda_{\max}\{C\Lambda C^{\mathrm{T}}C\Lambda C^{\mathrm{T}}\}.$$

It is easy to verify that $-C\Lambda C^{\mathrm{T}}$ is positive semi-definite, so there exists an orthogonal matrix T such that

$$TC\Lambda C^{\mathrm{T}}T^{\mathrm{T}}=\mathrm{diag}(\lambda_1,\ldots,\lambda_n).$$

Then it is easy to see that

$$\|C(sI-A)^{-1}C^{\mathrm{T}}\|_\infty^2=\max\{\lambda_1^2,\ldots,\lambda_n^2\}.$$

We assume that λ_1^2 is the biggest value among them. Then it is easy to verify that $y_0 = Te$ with $e = (1, 0, \ldots, 0)$ will satisfy

$$\| y_0^{\mathrm{T}} C(sI - A)^{-1} C^{\mathrm{T}} y_0 \|_\infty^2 = \lambda_1^2.$$

This completes the proof. □

Now, we are in a position to prove our main result.

Proof. From Lemma A2, we get

$$\| C(sI - A)^{-1} C^{\mathrm{T}} \|_\infty = \max_{y^{\mathrm{T}} y \leqslant 1} \| y^{\mathrm{T}} C(sI - A)^{-1} C^{\mathrm{T}} y \|_\infty .$$

Then, we obtain

$$\| y^{\mathrm{T}} C(sI - A)^{-1} C^{\mathrm{T}} y \|_\infty = - y^{\mathrm{T}} C A^{-1} C^{\mathrm{T}} y.$$

From Lemma A1, it is easy to obtain the main result. □

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